On roots of algebraic equations of the higher degree with real coefficients

Rakhshanda Dzhabarzadeh
Institute of Mathematics & Mechanic of NAS of Azerbaijan, department of functional analysis, Azerbaijan, Baku

Abstract. *The paper is devoted to researches of the problems, concerning nonlinear algebraic equations. This manner of investigation is by the new method of finding of roots of an algebraic nonlinear equation. In the case when algebraic system is linear and the number of variables coincides with the number of equations we have a fairly well-studied the system of algebraic equations.*

Many practical problems is reduced to the solution of algebraic equation of the higher degree. It is studed the algebraic equation of the higher degree in one unknown. To such equation we come at solving many practical problems. The author uses the methods and results of multiparameter system of selfadjoint operators in Hilbert space. In this article is given the kind of all modules of roots of algebraic equation.

Keywords: *root, equation, module, maximum*

1. Introduction

Linear algebra is central to modern mathematics and its applications . Linear algebra is a successful theory, its methods have been developed and generalized in other parts of mathematics. In module theory, one replaces the field of scalars by a ring. The concepts of linear independence, span, basis, and dimension (which are called the rank in module theory) still make sense.

An elementary application of linear algebra is to find the solution of a system of linear algebraic equations in several unknowns.

The study of linear algebra first emerged from the study of determinants, which were used to solve systems of linear equations. Determinants were used by Leibniz in 1693, and subsequently, Gabriel Cramer devised Cramer's Rule for solving linear systems in 1750. In this paper is offered the new approach of the decision of nonlinear algebraic equations. Naturally, the study of Linear Algebra includes the topics of vector algebra, matrix algebra, and the theory of vector spaces. It includes a range of theorems and applications in different branches of linear algebra, such as linear systems, matrices, operators, etc. Functional analysis studies the infinite-dimensional version of the theory of vector spaces. Techniques from linear algebra are also used in analytic geometry, engineering, in physics, in natural sciences, in computer science, on computer animation, and in the social sciences (particularly in economics).

The solution of many practical problems is reduced to the solution of various types of equations. The achievements in physics, engineering, information technologies only confirm this. In this article we study the algebraic equation

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0 \quad (1)$$

In the 1920, the Norwegian mathematician N. Abel proved that the roots of the equations (1) of the 5th and higher degrees cannot be expressed in terms of radicals. However, the roots of an n -th degree equation can be found with any reassigned accuracy using numerical methods. F. Whiet gave a full account of the problems connected with the solution of the equations of the third fourth degree. In 1545, the book of the Italian mathematician D. Cardano "Great Art, or the Rules of Algebra", where along with other questions of algebra, the general methods for solving cubic equations are considered, as well as the method of solving equations of the fourth degree discovered by his pupil L. Ferrari.

In the 16th century, Italian mathematicians succeeded in finding formulas for $n=3$ and $n=4.$ Simultaneously, the question of the general solution of equations of the third degree was dealt with by Scipio Dal Farro, his disciple Fiori and Tartaglia

It is known that the roots of equation (1) coincide with the eigenvalues of equation

$$(A+\lambda B)z = 0 \quad (2)$$

where the operator A (respectively, B) is given in space R^n by means of matrices

$$A = \begin{pmatrix} a_0 & a_1 a_n^{-\frac{1}{n}} & a_2 a_n^{-\frac{2}{n}} & \dots & a_{n-1} a_n^{-\frac{n-1}{n}} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \text{(respectively, } B = \begin{pmatrix} 0 & 0 & 0 & \dots & a_n^{\frac{1}{n}} \\ a_n^{\frac{1}{n}} & 0 & 0 & \dots & 0 \\ 0 & a_n^{\frac{1}{n}} & 0 & \dots & 0 \\ . & . & . & \dots & 0 \\ 0 & 0 & 0 & \dots a_n^{\frac{1}{n}} & 0 \end{pmatrix} \text{)}$$

$$\text{or } (A_1 + \lambda B_1) z = 0 \quad (3)$$

where $A_1 = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & a_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ . & . & . & \dots & 0 \\ 0 & 0 & 0 & \dots 1 & 0 \end{pmatrix}$

One can give many variants of equations in space R^n for which the eigenvalues coincide with the roots of equation (1).

$a_n \neq 0$ -real number. Using the results of the multiparameter theory in this paper, we establish the module of all the roots of the algebraic equation (1).

We give some definitions for multiparameter systems, which are necessary for understanding the further presentation of the material.

2. Known results of the spectral theory of selfadjoint multiparameter systems

For the purpose of presenting of the method of investigation of multiparameter spectral theory we present some results in this direction.

Let be

$$T_r^+ f_r + \sum_{s=1}^{n} \lambda_s V_{r,s}^+ f_r = 0;\ f_r \in H_r; r = 1,2,\dots,n \quad (4)$$

the multiparameter system of operators with n parameters $, H_1, H_2, \dots, H_n$ are separable Hilbert spaces. $H = H_1 \otimes H_2 \otimes \dots \otimes H_n$ is the tensor product of spaces $H_1, H_2, \dots, H_n$.

Operators $T_r, V_{r,s} : H_r \to H_r, s = 1,2,\dots,n$ are bounded and Hermitian in the space H_r For each set of elements $f_r \in H_r,\ f_r \neq 0, r = 1,2,\dots,n$ determinant $\det(V_{r,s} f_r, f_r) > 0$, where (.,.) is the inner product in H_r. [1],[2],[3].

Operators $\Delta_s ; H \to H, s = 0,1,\dots,n$ are defined as follows: let $f = f_1 \otimes f_2 \otimes \dots \otimes f_n$ be a decomposible tensor in H and $\alpha_0, \alpha_1, \dots, \alpha_n$ be an arbitrary complex numbers. Then $\Delta_0 f, \Delta_1 f, \dots, \Delta_n f$ are determined by equation (5)

$$\sum_{s=0}^{n} \alpha_s \Delta_s f = \otimes \begin{vmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_n \\ T_1 f_1 & V_{1,1} f_1 & \dots & V_{1,n} f_1 \\ . & . & \dots & . \\ T_n f_n & V_{n,1} & \dots & V_{n,n} f_n \end{vmatrix}$$

where the determinant can be extended to the entire space by linearity and continuity.

Δ_s is determined on the decomposable tensor from H if $\alpha_0 = \alpha_1 = \dots = \alpha_{s-1} = \alpha_{s+1} = \dots = \alpha_n ; \alpha_s = 1$ in the determinant and on all other elements of the space H is defined on linearity and continuity.

The inner product $[f, g]$ is given by the expression $(\Delta_0 f, g)$ where (f, g) is the inner product in space H .The norms induced by these inner products are equivalent, and thus topological concepts as continuity of the operators and the convergence of sequences of elements are equal with respect to these standards. In the future Γ_i will denote the operator $\Delta_0^{-1}\Delta_i (i = 1,2,\dots,n)$.

Theorem 1. [1], [6], [14]. Suppose. $D(\Delta_0^{-1}) \subset R(\Delta_i); i = 1,2,\dots,n$,then $\Gamma_i = \Gamma_i^* (i = 1,2,\dots,n).$

Proof. As composed of a Hermitian operator; for all we have

Suppose there is an operator from space and on certain decomposable tensor of the space with the help of the operator and the formula. On all other elements of the space operator defined by linearity and continuity. The operator called the induced operator in space.

Theorem 2 . [1],[6] $\Delta_i = \Delta_i^* (i = 0,1,\dots,n)$

Proof . All operators in the $\Delta_i (i = 0,1,\dots, n)$ are Hermitian and for any elements $f, g \in H;$ we have

$$[\Gamma_r f, g] = (\Delta_0 \Delta_0^{-1} \Delta_r f, g) = (f, \Delta_r g) = (f, \Delta_0 \Delta_0^{-1} \Delta_r g) = (\Delta_0 f, \Gamma_r g) = [f, \Gamma_r g]$$

Consequently Γ_i is selfadjont.

Operator A_i^+ is named by operator, induced to space H by A_i and is constructed by following: on decomposable tensor $f = f_1 \otimes f_2 \otimes \dots \otimes f_n$ we have

$A_r^+ f = f_1 \otimes f_2 \otimes \dots \otimes A_r f_r \otimes f_{r+1} \otimes \dots \otimes f_n$ and on other elements of H operator A_i^+

Is defined on linearity and continuity. The spectrum σ of the system $\{T_r, V_{r,s}\}$ is defined in [32] as a vehicle operator-valued measure Then σ there is a compact subset R^k of measures $\{E(M)f, g\}$ and indeed fade out σ_0. If $E(\lambda) = E(\lambda_1, \lambda_2, ..., \lambda_n)$ there is a Borel function defined on σ, and we can define $F(\Gamma) = F(\Gamma_1, ..., \Gamma_n)$ the operator as follows:

$i) DF(\Gamma) = \{F \in H / \int_\sigma |F(\lambda)|^2 [E(d\lambda) f, f] < \infty\}$

$(ii) f \in D(F(\Gamma))$

and to have $[F(\Gamma) f, g] = \int_\sigma F(\lambda)[E(d\lambda) f, g]$ for an arbitrary $g \in H$ (6)

If $F(\lambda)$ is a bounded function, then $DF(\Gamma) = H$ If $F(\lambda)$ unlimited and has a dense domain, the details of these results can be found in E. Prugovečku. [13]

If the space is of finite dimension, then, naturally, the integral in (6) becomes a sum. As a consequence of the theorem, there is a separation of the parameters. If $\tilde{z} = x_1 \otimes x_2 \otimes ... x_n$ is an eigenvector of the system (4) with an eigenvalue $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, then the equalities $\Delta_0^{-1} \Delta_i x = \lambda_i x \qquad (i = 1, 2, ..., n)$ hold.

Let us return to our equation (1). We introduce the notation: $x^i = \lambda_i \ (i = 1, 2, ..., n)$

Then our equation (1) takes the form

$a_0 + \lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n = 0$ (7)

We supplement equation (7) with the help of the new equations,

$$(\lambda_{i-1} t_1 + \lambda_i t_0 + \lambda_{i+1} t_2) z_i = 0$$
$$i = 1, 2, ..., n-1; \ \lambda_0 = 1 \qquad (8)$$

compiled in such a way that the connections between the parameters would not be violated in equation (1). Then we get a multiparameter system with n equations and n parameters

$$a_0 + \lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n = 0$$
$$(\lambda_{i-1} t_1 + \lambda_i t_0 + \lambda_{i+1} t_2) z_i = 0 \qquad (9)$$
$$i = 1, 2, ..., n-1; \ \lambda_0 = 1; \ z_i \in R^2$$

where $z_i \in R^2$, t_0, t_1, t_2 are selfadjoint operators, defined with help the matrices

$$t_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ t_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For the multiparameter system (9) we build the operator

$$\Delta_0 = \begin{pmatrix} a_1^+ & a_2^+ & a_3^+ & ... & a_n^+ \\ t_0^+ & t_2^+ & 0 & ... & 0 \\ t_1^+ & t_0^+ & t_2^+ & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ...t_1^+ t_0^+ & t_2^+ \end{pmatrix}$$

The operator is obtained by replacing the $i-$ column by a column of free terms $(-a_0, -t_1^=, 0, ..., 0)$

All operators Δ_i are self-adjoint and act in finite-dimensional space $R^{2^{n-1}}$.

Theorem 3. Let be, $a_i \ (i = 0, 1, ..., n)$, are real numbers, $a_0 \neq 0, \ a_n \neq 0$, $Ker \Delta_0 = \{0\}$ then the module all roots of equation (1) are defined by formulae $|\lambda_i| = \max \dfrac{(\Delta_i x, x)}{(\Delta_0 x, x)}$, $x \in R^{2^{n-1}} \quad (i = 1, ..., n)$

Proof. If the eigenvalue of the multiparameter system (9) is an eigenvalue $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, then the following equalities $\Delta_0^{-1} \Delta_i x = \lambda_i x \qquad (i = 1, 2, ..., n)$ hold

If $a_0 \neq 0$, then equation (1) has not the root equally to 0. Indeed, let be $\bar{z} \neq \theta$

$(t_1 + \lambda_1 t_0 + \lambda_2 t_2)z = 0$ and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

We have

$$z_1 + \lambda_1 z_2 = 0$$
$$\lambda_1 z_1 + \lambda_2 z_2 = 0 \qquad (10)$$

$\lambda_2 = \lambda_1^2,\ \lambda_2 \neq 0$.,

By analogue from

$$(\lambda_{i-1} t_1 + \lambda_i t_0 + \lambda_{i+1} t_2) z_i = 0$$
$$i = 1, 2, \ldots, n-1;\ \lambda_0 = 1;\ z_i \in R^2$$

follows $\lambda_i = \lambda_1^i,\ \lambda_i \neq 0$. If we take into account the adopted notation, then

$$\Delta_0^{-1}\Delta_i \tilde{z} = \lambda_1^i \tilde{z} = x^i \tilde{z} \qquad (i = 1, 2, \ldots, n)$$

where x the eigenvector of the system (1). The operators $\Delta_0^{-1}\Delta_i$ are self-adjoint if an inner product is given in the form $[x, y] = (\Delta_0 x, y)$ -when $(\Delta_0 x, y)$ inner product in the traditional sense. Then

$x^i = \lambda_i\ (i = 1, 2, \ldots, n)$,

We use the known facts from spectral theory of operators in Hilbert space. [15]

The maximum vector e of the bounded operator in a Hilbert space is the unit vector e, when the magnitude $\|Ae\|$ reaches its maximum value $\|A\| = M$. In Hilbert space completely continuous symmetric operator has the maximum vector.

$$\|Ae\| = \|A\| = M, \qquad \|Ae\|^2 = \|A\|^2 = M^2, = \qquad \|Ae\|^2 = \|A^2 e\| = (Ae, Ae)$$

If $\|e\| = 1$, and A is a symmetric operator, then $\|Ae\|^2 \leq \|A^2 e\|$

moreover, the equality sign is possible if and only if there is an eigenvector e of A^2 with an eigenvalue $\|Ae\|^2$. If the operator A^2 has eigenvalue M^2, the operator A has an eigenvector with its eigenvalue of M or $-M$.

$$[\Delta_0^{-1}\Delta_1 \tilde{z}, \tilde{z}] \leq c[\tilde{z}, \tilde{z}] \quad (\Delta_1 \tilde{z}, \ddot{z}) \leq c(\Delta_0 \tilde{z}, \tilde{z}) \quad (\Delta_1 \tilde{z}, \ddot{z})(\Delta_0 \tilde{z}, \tilde{z})^{-1} \leq c$$

$$\max(\Delta_1 \tilde{z}, \ddot{z})(\Delta_0 \tilde{z}, \tilde{z})^{-1} = |\lambda_1|$$

$$\max|\lambda_i| = \max \frac{(\Delta_i x, x)}{(\Delta_0 x, x)} \quad , x \in R_i^{2^{n-1}} \quad (i = 1, \ldots, n)$$

. Now let R_1 be the subspace spanned by the maximal vector X on which it is attained

$\max \frac{(\Delta_i x, x)}{(\Delta_0 x, x)}\ , x \in R_i^{2^{n-1}}\ (i = 1, \ldots, n)$. On the orthogonal complement of the subspaces R_1 to the space $R^{2^{n-1}}$ expression $\max \frac{(\Delta_i x, x)}{(\Delta_0 x, x)}\ , x \in R^{2^{n-1}} \Theta\ R_1$

defines the second largest root modulus of equation (1). The moduli of all the roots of the algebraic equation (1) are defined similarly. The Theorem 3 is proved.

References:

1. Atkinson F.V. Multiparameter spectral theory. Bull.Amer.Math.Soc.1968, 74, 1-27

2. Atkinson F.V. Multiparameter eigenvalue problems: Matrices and compact operators. Academic Press: New York, London, 1972

3. Berezansky Y.M. Expansion in eigenfunctions selfadjoint operators in Hilbert space. Scientific notes of the Moscow State University, 1951, vol. 148, v.4, pp.69-107. Naukova Dumka(in Russian).

4. Binding P. Multiparameter definiteness conditions. Proc. Roy. Soc.Edinburgh, 89 A, 1981, 319-332.

5. Binding P., Browne P.J.,Tyrin L. Existence conditions for eigenvalue problems generated by compact multiparameter operator. Proc.Royal Soc. Edinburgh, 96A, №3-4, 1984, 261-274.

6. Browne P.J. Multiparameter spectral theory. Indiana Univ. Math. J, №24, №3, 1974.

7. Dzhabarzadeh R.M. About solutions of nonlinear algebraic system with two variables. Pure and Applied Mathematics Journal, vol. 2, No. 1, pp. 32-37, 2013

8. Dzhabarzadeh R.M. Multiparameter spectral theory. Lambert Academic Publishing, 2012, p. 184 (in Russian)

9. Dzhabarzadeh R.M. Nonlinear algebraic system of equations. Lambert Academic Publishing, 2013, p. 101 (in Russian)

10. Dzhabarzadeh R.M. Nonlinear algebraic system with three unknowns variables. International Journal of Research Engineering and Science (IJRES), Volume2, Issue6,14 June,2014,pp 54-59

11. Dzhabarzadeh R.M. , Salmanova G.H. Multiparameter system of operators, not linearly depending on parameters. American Journal of Mathematics and Mathematical Sciences.-2012, vol.1, №1, pp.32-37, 2013

12. Huseyin Tevfik Pasha .Linear algebra(a second edition, revised and enlarged in,1892, Istambul) Constantinople: A.N.Boyajian

13. Prugoveĉ ku E. Quantum mechanics in Hilbert space. Academic Press, New York, London, 1971.

14. Sleeman B.D. Multiparameter spectral theory in Hilbert space. J.Math-Anal.Appl. 1978, 65, №3,pp. 511-530.

15. Shilov G.E. Mathematical analyses, M.,1960 .